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A CALCULUS OF NATURAL DEDUCTIONS FOR THE FULL FIRST-ORDER PREDICATE LOGIC WITH IDENTITY

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Natural deductions form an important tool in applications of logic to scientific theories. Our calculus for natural deductions is formulated in such a manner that it can be applied to the language of the full first-order predicate logic. Among its features are a certain symmetry of its deduction rules and simplified restrictions governing finished deductions. The adequacy of our natural deduction system is established by means of showing its equivalence with a more standard type of deduction system, known to be sound and complete. The proof for the equivalence of the two systems is constructive so that any deduction in one of the systems provides a deduction in the other system.

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INTRODUCTION

From a semantic standpoint, a logical calculus may be viewed as a means of generating the set of all consequence relations by an algorithmic procedure. Gödel's Completeness Theorem established the existence of such calculi for first-order predicate logic. Today many logical calculi are known which generate exactly the set of all consequence relations for first-order predicate logic. The choice among these calculi depends in part on their intended use. For applications to scientific theories such as mathematics, it will be desirable, especially in view of the undecidability of predicate logic, to have a calculus which provides simple, convenient tools for the derivation of consequences. In particular, for the working mathematician, this leads to the search for calculi whose deduction rules resemble as closely as possible the familiar mathematical proof procedures. Several such calculi, known as "natural deduction" systems, have been developed for more or less restricted first-order predicate logics, among others by Gentzen (1934) and Quine (1950). We present here a natural deduction system N which is formulated for the full first-order predicate logic with identity, including free individual variables, individual constants, and functional variables. Among the features of our natural deduction system N are the symmetry of its deduction rules as well as the simplicity of the restrictions governing finished deductions. Our natural deduction system N is shown to be sound and strongly complete in the sense that it yields exactly the set of consequence relations of the full first-order predicate logic with identity.

THE FORMAL LANGUAGE OF THE FULL FIRST-ORDER PREDICATE LOGIC

The *vocabulary* for the full first-order predicate logic contains (i) a denumerable set of individual variables, (ii) a countable (i.e., finite or denumerable) set of individual constants, (iii) for each integer $n>0$ a countable set of n-ary functional variables, (iv) for each integer $n\geqslant 0$ a countable set of n-ary predicate variables, (v) the identity symbol $\equiv$, (vi) the propositional connectives $\sim$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, (vii) the quantifiers $\forall$ and $\exists$, and (viii) the parentheses (,).

The set of *terms* is the smallest set which contains the individual variables and the individual constants, and which with any n-ary functional variable f and any n terms $t_1, \ldots, t_n$ also contains $ft_1 \ldots t_n$. *Atomic* formulas are the 0-ary predicate variable, and all expressions of the form $pt_1 \ldots t_n$ where p is any n-ary predicate variable and $t_1, \ldots, t_n$ are any terms, and all expressions of the form $t_1 \equiv t_2$ where t_1 and t_2 are any terms.

Formulas are defined inductively by the following conditions:

(1) Each atomic formula is a formula.

(2) If B is a formula, then $(\sim B)$ is a formula.

(3) If B and C are formulas, then $(B \wedge C)$, $(B \vee C)$, $(B \rightarrow C)$ and $(B \leftrightarrow C)$ are formulas.

(4) If B is any formula and x is any individual variable, then $(\forall xB)$ and $(\exists xB)$ are formulas.

The notion of *free* occurrence of a term t in a formula A can be described inductively according to the inductive definition of formula as follows:

(1) Any occurrence of a term t in an atomic formula is a free occurrence.

(2) If t occurs free in the formula B, then t occurs free in the formula ($\sim$ B).

(3) If t occurs free in the formula B or in the formula C, then t occurs free in the formulas ($B \wedge C$), ($B \vee C$), ($B \rightarrow C$) and ($B \leftrightarrow C$).

(4) If t occurs free in the formula B and x is an individual variable not occurring in t, then t occurs free in the formulas ($\forall x B$) and ($\exists x B$).

If A is any formula, x any individual variable and t any term, and there exists a formula which is the result of replacing in A each free occurrence of x by a free occurrence of t, then B is said to be obtained by a *free substitution* of t for x in A, abbreviated: Subst A x/t B. For given A, x, and t there is at most one formula B such that Subst A x/t B. If Subst A x/t B, then this unique formula is also indicated by $A[x/t]$.

If S is any set of formulas, then Subst S x/t S′ indicates that for each formula A in S there is a formula A′ with Subst A x/t A′ and S′ consists of all these formulas A′. Again, if Subst S x/t S′, we shall indicate S′ also by $S[x/t]$.

We note the following two properties regarding free substitutions: (a) If y is any individual variable not occurring in the formula A, then there is a unique formula B such that Subst A x/y B and Subst B y/x A. (b) If A, B, C are any formulas, x and y are any individual variables with y not free in A, and t is any term such that Subst A x/y B and Subst B y/t C, then Subst A x/t C.

THE NATURAL DEDUCTION SYSTEM *N*

A *natural deduction* in the system N is a finite sequence Σ of ordered pairs $\langle S_k, A_k \rangle$, $1 \leqslant k \leqslant n$ for some positive integer n, where S_k is a (possibly empty) set of formulas, namely the set of assumptions upon which the formula A_k depends (according to the regulations stated below). Each pair $\langle S_k, A_k \rangle$ of Σ must satisfy (at least) one of the following conditions (also called the deduction rules of the calculus):

Assumption Introduction (AI): $S_k = \{A_k\}$.

Assumption Elimination (AE): There is $i < k$ and a formula B such that $S_k = S_i - \{B\}$ and $A_k = B \rightarrow A_i$.

Tautological Inference (TI): Either $S_k = \emptyset$ and A_k is tautologous, or there exist $i_1 < k, \ldots, i_m < k$ such that $S_k = S_{i_1} \cup \ldots \cup S_{i_m}$ and $(A_{i_1} \wedge \ldots \wedge A_{i_m}) \rightarrow A_k$ is tautologous.

$\forall$-*Elimination* ($\forall$E): There is $i < k$, a formula B, an individual variable x and a term t such that $S_k = S_i$, $A_i = \forall x B$ and $A_k = B[x/t]$.

$\forall$-*Introduction* ($\forall$I): There is $i < k$, a formula B and individual variables x and y such that $S_k = S_i$, $A_i = B[x/y]$, $A_k = \forall x B$ and $B[x/y]\ [y/x] = B$. The individual variable y is to be *marked* in dependence of the individual variables occurring free in $\forall x B$.

$\exists$-*Elimination* ($\exists$E): There is $i < k$, a formula B and individual variables x and y such that $S_k = S_i$, $A_i = \exists x B$, $A_k = B[x/y]$ and $B[x/y]\,[y/x] = B$. The individual variable y is to be *marked* in dependence of the individual variables occurring free in $\exists x B$.

$\exists$-*Introduction* ($\exists$I). There is $i < k$, a formula B, an individual variable x and a term t such that $S_k = S_i$, $A_i = B[x/t]$ and $A_k = \exists x B$.

Identity Elimination (IE): There is $i < k$, a formula B, a term t and an individual variable x not occurring in t such that $S_k = S_i$, $A_i = \forall x(x \equiv t \rightarrow B)$ and $A_k = B[x/t]$.

Identity Introduction (II): There is $i < k$, a formula B, a term t and an individual variable x not occurring in t such that $S_k = S_i$, $A_i = B[x/t]$ and $A_k = \forall x(x \equiv t \rightarrow B)$.

A natural deduction Σ is said to be a *finished* deduction provided:

(i) No individual variable is marked more than once in Σ.
(ii) No individual variable marked in Σ occurs free in the formulas of the last pair of Σ.
(iii) The marked individual variables of Σ are not circularly dependent, i.e., the marked individual variables of Σ can be put in a sequence such that in this sequence no marked individual variable depends upon another marked individual variable occurring later in this sequence (a sequence with this property will be called a *normed ordering* of the marked individual variables of Σ).

A formula A is *deducible* from a set S of formulas in the system N, abbreviated: $S \rhd A$, if and only if there is a finished natural deduction Σ whose last pair has the form $\langle S', A \rangle$ where $S' \subseteq S$.

The condition (TI) for tautological inferences is semantic in nature, but could be replaced in a variety of ways by purely syntactical conditions. For example, appropriate conditions which provide for the introduction and elimination of propositional connectives as in Schneider (1973), could serve the purpose of condition (TI). However, since the truth-table method provides such a simple means of testing whether or not a formula is tautologous, it appears that for practical purposes our condition (TI) is preferable.

Each set S_k of assumptions occurring in any pair $\langle S_k, A_k \rangle$ of a natural deduction Σ is finite since each formula of S_k must have been introduced in Σ originally on account of condition (AI), and Σ itself is a finite sequence. Thus, $S \mathrel{|\!\!>} A$ if and only if there is a *finite* subset S' of S such that $S' \mathrel{|\!\!>} A$.

In practice, it will be convenient to number the ordered pairs $\langle S_k, A_k \rangle$ of a natural deduction Σ consecutively and to represent the formulas of the set S_k of assumptions by the numbers of the pairs in which they are first introduced according to condition (AI). A typical k-th pair $\langle S_k, A_k \rangle$ of Σ with $S_k = \{B_1, \ldots, B_r\}$ can then be represented by the line

$$k\ \{k_1, \ldots, k_r\} \qquad A_k$$

where the lines $k_i \leqslant k$, $i = 1, \ldots, r$, have the form

$$k_i\ \{k_i\} \qquad B_i$$

If in accordance with conditions (∀I) or (∃E) an individual variable y is to be marked in dependence of individual variables $z_1, \ldots, z_i$, we shall indicate this by writing $y(z_1, \ldots, z_i)$ to the left of the corresponding line.

The restrictions imposed on finished deductions with regard to marked individual variables are necessary in order to preserve soundness of our system *N*. The following four deductions show the necessity of the restrictions; in each example just one of the restrictions is violated, resulting in a relationship which is not a consequence relation.

(a)	Natural deduction:	Comment:
	1{1} ∃xPx	(AI)
y	2{1} Py	(∃E), 1
	3{ } ∃xPx → Py	(AE), 2

In this deduction the marked individual variable y occurs free in the formula of the last line, in violation of restriction (ii) of a finished deduction. Note that ∃xPx → Py is not a valid formula.

(b)	Natural deduction:	Comment:
	1{1} Py	(AI)
y	2{1} ∀xPx	(∀I), 1

In this deduction the marked individual variable y occurs free in the assumption upon which the formula of the last line depends, in violation of restriction (ii) of a finished deduction. Note that ∀xPx is not a consequence of Py.

(c)	Natural deduction:	Comment:
	1{1} ∃xPx	(AI)
y	2{1} Py	(∃E), 1
y	3{1} ∀xPx	(∀I), 2
	4{ } ∃xPx → ∀xPx	(AE), 3

In this deduction, the individual variable y is marked twice in violation of restriction (i) of a finished deduction. Note that the formula ∃xPx → ∀xPx is not valid.

(d)	Natural deduction:	Comment:
	1{1} ∀y∃xPxy	(AI)
	2{1} ∃xPxu	(∀E), 1
v(u)	3{1} Pvu	(∃E), 2
u(v)	4{1} ∀yPvy	(∀I), 3
	5{1} ∃x∀yPxy	(∃I), 4
	6{ } ∀y∃xPxy → ∃x∀yPxy	(AE), 5

In this deduction the marked individual variables u and v are circularly dependent, in violation of restriction (iii) of a finished deduction. Note that the formula ∀y∃xPxy → ∃x∀yPxy is not valid.

In the following deductions all restrictions concerning finished natural deductions are observed. Each of the stated deducibility relation is indeed also a consequence relation.

(1) {∃x∀yPxy} |> ∀y∃xPxy

	Natural deduction:	Comment:
	1{1} ∃x∀yPxy	(AI)
u	2{1} ∀yPuy	(∃E), 1
	3{1} Puv	(∀E), 2
	4{1} ∃xPxv	(∃I), 3
v	5{1} ∀y∃xPxy	(∀I), 4

(2) {∀x∃yPxy, ∀x∀y(Pxy → Pyx), ∀x∀y∀z(Pxy ∧ Pyz → Pxz)} |> ∀xPxx

	Natural deduction:	Comment:
	1{1} ∀x∃yPxy	(AI)
	2{2} ∀x∀y(Pxy → Pyx)	(AI)
	3{3} ∀x∀y∀z(Pxy ∧ Pyz → Pxz)	(AI)
	4{1} ∃yPuy	(∀E), 1
v(u)	5{1} Puv	(∃E), 4
	6{2} ∀y(Puy → Pyu)	(∀E), 2
	7{2} Puv → Pvu	(∀E), 6

	8(1,2) Pvu	(TI), 5, 7
	9(3) $\forall y \forall z(Puy \wedge Pyz \to Puz)$	($\forall$E), 3
	10(3) $\forall z(Puv \wedge Pvz \to Puz)$	($\forall$E), 9
	11(3) $Puv \wedge Pvu \to Puu$	($\forall$E), 10
	12(1,2,3) Puu	(TI), 5, 8, 11
u	13(1,2,3) $\forall x Pxx$	($\forall$I), 12

(3) $\rhd \forall x \exists y\ x{\equiv}y$

	Natural deduction:	Comment:
	1(1) $u{\equiv}v$	(AI)
	2(1) $\exists y\ u{\equiv}y$	($\exists$I), 1
	3( ) $u{\equiv}v \to \exists y\ u{\equiv}y$	($\forall$I), 3
u(v)	4( ) $\forall u(u{\equiv}v \to \exists y\ u{\equiv}y)$	($\forall$I), 3
	5( ) $\exists y\ v{\equiv}y$	(IE), 4
v	6( ) $\forall x \exists y\ x{\equiv}y$	($\forall$I), 5

SECONDARY DEDUCTION RULES FOR THE SYSTEM *N*

If Σ is a natural deduction and v any individual variable not occurring in Σ, we denote by $\Sigma[u/v]$ the result of replacing in Σ each formula B by the formula $B[u/v]$ and each occurrence of u as a marked individual variable by v. By induction, on the lines of Σ one shows

Lemma 1: If Σ is a finished deduction for $S \rhd A$ and v is any individual variable not occurring in Σ, then $\Sigma[u/v]$ is a finished deduction for $S[u/v] \rhd A[u/v]$. Moreover, if $<y_1, \ldots, y_r>$ is a normed ordering of the marked individual variables of Σ, and $z_i = y_i$ for $y_i \neq u$ whereas $z_i = v$ for $y_i = u$, then $<z_1, \ldots, z_r>$ is a normed ordering of the marked individual variables of $\Sigma[u/v]$.

If u is a marked individual variable of a natural deduction Σ which is finished, then u does not occur free in the last line of Σ; hence, in $\Sigma[u/v]$ the last line is the same as that of Σ. Thus we have

Lemma 2: If Σ is a finished deduction for $S \rhd A$, u is any individual variable marked in Σ, and v is any individual variable not occurring in Σ, then $\Sigma[u/v]$ is a finished deduction for $S \rhd A$; moreover, in $\Sigma[u/v]$ the individual variable v is marked in place of u.

Applying this lemma consecutively for each of the marked individual variables of a finished natural deduction, we obtain

Lemma 3: If Σ is a finished deduction for $S \rhd A$ with $<y_1, \ldots, y_r>$ as a normed ordering of the marked individual variables of Σ, and $z_1, \ldots, z_r$ are distinct individual variables not occurring in Σ, then $\Sigma' = \Sigma[y_1/z_1] \ldots [y_r/z_r]$ is a finished deduction for $S \rhd A$ with $<z_1, \ldots, z_r>$ as a normed ordering of the marked individual variables of Σ.

In view of this last lemma, it is clear that two or more finished deductions can be combined after appropriate relabeling so as to form a new finished deduction. For example, if Σ_1 and Σ_2 are two finished deductions for $S_1 \rhd A_1$ and $S_2 \rhd A_2$, respectively, with $<y_1, \ldots, y_r>$ as a normed ordering of the marked individual variables of Σ_1 and $<z_1, \ldots, z_s>$ as a normed ordering of the marked individual variables of Σ_2, then these two deductions can be combined by first replacing Σ_2 by $\Sigma' = \Sigma_2[z_1/v_1] \ldots [z_s/v_s]$ where $v_1, \ldots, v_s$ are individual variables not occurring in Σ_1 and Σ_2, and then adjoining Σ' to Σ_1. The resulting sequence Σ^* is again a finished deduction having $<y_1, \ldots, y_r, z_1, \ldots, z_s>$ as a normed ordering of the marked variables of Σ^*. Moreover, since the last pair of Σ_1 is of the form $<S_1', A_1>$ with $S_1' \subseteq S_1$ and the last pair of Σ_2 is of the form $<S_2', A_2>$ with $S_2' \subseteq S_2$, we can adjoin to Σ^* the pair $<S_1' \cup S_2', A_1 \wedge A_2>$ on account of condition (TI); the resulting sequence is clearly a finished deduction for $S_1 \cup S_2 \rhd A_1 \wedge A_2$. In this manner one shows

Lemma 4: If $S_1 \rhd A_1, \ldots, S_k \rhd A_k$, then $S_1 \cup \ldots \cup S_k \rhd A_1 \wedge \ldots \wedge A_k$.

On the basis of these lemmas, we shall now derive several secondary deduction rules for our system *N*. In the proofs that follow, we shall represent the k-th ordered pair $<(B_1, \ldots, B_t), C>$ of a deduction Σ simply as the line

$$k(B_1, \ldots, B_t) \quad C$$

(AI) If $A \in S$ then $S \rhd A$.

This secondary deduction rule follows at once from the one line deduction: 1(A) A.

(AE) If $S \rhd A$ then $S - (B) \rhd B \to A$.

Let Σ be a finished deduction for $S \rhd A$ whose last line is

$$n(A_1, \ldots, A_t) \quad A \qquad \text{with } A_1, \ldots, A_t \in S.$$

We continue this deduction as follows:

n+1 (B)	B	(AI)
n+2 $(A_1, \ldots, A_t, B)$	A	(TI), n, n+1

$$n{+}3\{A_1, \ldots, A_t\} \quad B \to A \quad (AE), n{+}2$$

These n+3 lines constitute a finished deduction for $S - \{B\} \mathrel{|\!\!>} B \to A$.

(TA) If A is a tautologous formula, then $\mathrel{|\!\!>} A$.

Suppose A is tautologous, then by (TI) we get the one-line finished deduction: $1\{\ \}$ A for $\mathrel{|\!\!>} A$.

(TI) If $S_1 \mathrel{|\!\!>} A_1, \ldots, S_m \mathrel{|\!\!>} A_m$, and $A_1 \wedge \ldots \wedge A_m \to A$ is tautologous, then $S_1 \cup \ldots \cup S_m \mathrel{|\!\!>} A$.

In view of Lemma 4, we obtain from $S_1 \mathrel{|\!\!>} A_1, \ldots, S_m \mathrel{|\!\!>} A_m$ at once $S_1 \cup \ldots \cup S_m \mathrel{|\!\!>} A_1 \wedge \ldots \wedge A_m$. Let Σ be a finished deduction for $S_1 \cup \ldots \cup S_m \mathrel{|\!\!>} A_1 \wedge \ldots \wedge A_m$ whose last line is

$n\{H_1, \ldots, H_t\}$ $A_1 \wedge \ldots \wedge A_m$ with $H_1, \ldots, H_t \in S_1 \cup \ldots \cup S_m$.

We continue this deduction as follows:

$$n{+}1\{H_1, \ldots, H_t\} \quad A \quad (TI)$$

This line is justified, since by assumption $A_1 \wedge \ldots \wedge A_m \to A$ is a tautologous formula. Thus these n+1 lines constitute a finished deduction for $S_1 \cup \ldots \cup S_m \mathrel{|\!\!>} A$.

(FS) If $S \mathrel{|\!\!>} A$, Subst A x/t B and x is not free in S, then $S \mathrel{|\!\!>} B$.

Proof: Suppose $S \mathrel{|\!\!>} A$, Subst A x/t B, and x is not free in S. Let Σ be a finished deduction for $S \mathrel{|\!\!>} A$ whose last line is thus of the form:

$$n\{H_1, \ldots, H_t\} \quad A \quad \text{with } H_1, \ldots, H_t \in S.$$

Let z be any individual variable not occurring in Σ. By Lemma 1, if we replace in Σ each free x by z, we obtain a finished deduction Σ' whose last line has the form

$$n\{H_1, \ldots, H_t\} \quad A' \quad \text{where Subst A x/z } A'.$$

Note that by this free substitution $H_1, \ldots, H_t$ are unchanged since x is not free in S and hence not free in $H_1, \ldots, H_t$. We continue this deduction Σ' as follows:

$$z(z_1, \ldots, z_r) \quad n{+}1\{H_1, \ldots, H_t\} \quad \forall zA' \quad (VI), n$$

where z is marked in dependence of the individual variables $z_1, \ldots, z_r$ occurring free in $\forall zA'$.

$$n{+}2\{H_1, \ldots, H_t\} \quad A'' \quad (VE), n{+}1$$

where Subst A' z/t A''. These n+2 lines constitute a finished deduction for $S \mathrel{|\!\!>} A''$. z is the only individual variable marked in addition to those individual variables already marked in Σ'. The choice of z guarantees that the conditions for a finished deduction are still met. Observe now that from Subst A x/z A', Subst A' z/t A'' and z not in A, we get Subst A x/t A''. Since, on the other hand, Subst A x/t B, we must have $A'' = B$ and, hence, $S \mathrel{|\!\!>} B$.

(AG) If $S \mathrel{|\!\!>} A \to B$ then $S \mathrel{|\!\!>} \forall xA \to B$

Proof: Let Σ be a finished deduction for $S \mathrel{|\!\!>} A \to B$ having as its last line

$$n\{H_1, \ldots, H_t\} \quad A \to B \quad \text{with } H_1, \ldots, H_t \in S$$

We continue this deduction as follows:

$n{+}1\{\forall xA\}$	$\forall xA$	(AI)
$n{+}2\{\forall xA\}$	A	(VE), n+1
$n{+}3\{H_1, \ldots, H_t, \forall xA\}$	B	(TI), n, n+2
$n{+}4\{H_1, \ldots, H_t\}$	$\forall xA \to B$	(AE), n+3

These n+4 lines constitute a finished deduction for $S \mathrel{|\!\!>} \forall xA \to B$.

(CG) If $S \mathrel{|\!\!>} A \to B$ and x is not free in $S \cup \{A\}$, then $S \mathrel{|\!\!>} A \to \forall xB$.

Proof: Suppose $S \mathrel{|\!\!>} A \to B$ and x not free in $S \cup \{A\}$. In view of Lemma 2, there exists a finished deduction Σ for $S \mathrel{|\!\!>} A \to B$ such that the individual variable x is not marked in Σ. The last line of Σ has the form

$$n\{H_1, \ldots, H_t\} \quad A \to B \quad \text{where } H_1, \ldots, H_t \in S.$$

We continue this deduction as follows:

	$n{+}1\{A\}$	A	(AI)
	$n{+}2\{H_1, \ldots, H_t, A\}$	B	(TI), n, n+1
$x(z_1, \ldots, z_r)$	$n{+}3\{H_1, \ldots, H_t, A\}$	$\forall xB$	(VI), n+2

where x is marked in dependence of the individual variables $z_1, \ldots, z_r$ occurring free in $\forall xB$.

$$n{+}4\{H_1, \ldots, H_t\} \quad A \to \forall xB \quad (AE), n{+}3$$

These n+4 lines provide a finished deduction Σ' for $S \mathrel{|\!\!>} A \rightarrow \forall x B$.

If $\langle y_1, \ldots, y_m \rangle$ is a normed ordering of the marked individual variables of Σ, then $\langle x, y_1, \ldots, y_m \rangle$ is a normed ordering of the marked individual variables of the augmented deduction Σ'; clearly x does not depend on $y_1, \ldots, y_m$ since $y_1, \ldots, y_m$ cannot occur free in the last line of Σ and, hence, cannot occur free in B.

In an analogous manner one can establish the following two secondary deduction rules:

(AP) If $S \mathrel{|\!\!>} A \rightarrow B$ and x is not free in $S \cup \{B\}$, then $S \mathrel{|\!\!>} \exists x A \rightarrow B$.

(CP) If $S \mathrel{|\!\!>} A \rightarrow B$, then $S \mathrel{|\!\!>} A \rightarrow \exists x B$.

We show next:

(II) If $S \mathrel{|\!\!>} B$, Subst A x/t B and x does not occur in t, then $S \mathrel{|\!\!>} \forall x(x \equiv t \rightarrow A)$.

Indeed, let Σ be a finished deduction for $S \mathrel{|\!\!>} B$ having as its last line

$$n\{H_1, \ldots, H_t\} \quad B \quad \text{where } H_1, \ldots, H_t \in S.$$

Assuming that Subst A x/t B and x is not occurring in t, we can continue this deduction by adding the line

$$n+1\{H_1, \ldots, H_t\} \quad \forall x(x \equiv t \rightarrow A)$$

on account of condition (II) of a natural deduction. These n+1 lines constitute a finished deduction for $S \mathrel{|\!\!>} \forall x(x \equiv t \rightarrow A)$.

Analogously one establishes the secondary deduction rule:

(IE) If $S \mathrel{|\!\!>} \forall x(x \equiv t \rightarrow A)$, Subst A x/t B and x does not occur in t, then $S \mathrel{|\!\!>} B$.

THE ADEQUACY OF THE SYSTEM $\mathcal{N}$

The secondary deduction rules, established in the preceding chapter, form the basis for a deduction system $\mathcal{S}$, described in Schneider (1976). Indeed, if we replace in our secondary deduction rules the deduction symbol $\mathrel{|\!\!>}$ by the symbol $\vdash$, we obtain the primary deduction rules of the system $\mathcal{S}$. A *deduction* Δ in this system $\mathcal{S}$ can then be described as a finite sequence $S_1 \vdash A_1, \ldots, S_n \vdash A_n$, where each element $S_k \vdash A_k$ of this sequence can either be justified on account of the rules (AI) or (TA), or else is obtainable from preceding elements of the sequence by means of one of the rules (AE), (TI), (FS), (AG), (CG), (AP), (CP), (II), (IE). A formula A is *deducible* from a set S of formulas in the system $\boldsymbol{S}$, abbreviated $S \vdash A$, if and only if there is a deduction Δ in the system $\mathcal{S}$ whose last element is of the form $S' \vdash A$ with $S' \subseteq S$.

Since each primary deduction rule of the system $\mathcal{S}$ has a corresponding secondary deduction rule in the system $\mathcal{N}$, it follows at once that each step of a deduction in the system $\mathcal{S}$ can be copied by a corresponding deduction in the system $\mathcal{N}$. Hence we get at once the

Theorem: If $S \vdash A$ then $S \mathrel{|\!\!>} A$.

Before showing the converse of this theorem, we remark that the rule (FS) can be strengthened as follows:

(FS*) If $S \vdash A$, Subst S x/t S′ and Subst A x/t A′, then $S' \vdash A'$.

A proof of this rule can be found in Schneider (1973); we shall use this rule in proving the next

Theorem: If $S \mathrel{|\!\!>} A$ then $S \vdash A$.

Proof: Let $\langle y_1, \ldots, y_r \rangle$ be a normed ordering of the marked individual variables in a finished natural deduction Σ of $S \mathrel{|\!\!>} A$. With each marked individual variable y_k, $k = 1, \ldots, r$, we associate a formula C_k as follows: If y_k was marked in Σ on account of an application of ($\forall$I), say upon proceeding from B_k to $\forall x_k A_k$ where Subst A_k x_k/y_k B_k and Subst B_k y_k/x_k A_k, then let $C_k = B_k \rightarrow \forall x_k A_k$. On the other hand, if y_k was marked in Σ on account of an application of ($\exists$E), say upon proceeding from $\exists x_k A_k$ to B_k where Subst A_k x_k/y_k B_k and Subst B_k y_k/x_k A_k, then let $C_k = \exists x_k A_k \rightarrow B_k$. Finally, let $C = C_1 \wedge \ldots \wedge C_r$. By induction on the lines of Σ, we now show easily: If in Σ line m has the form: $m\{B_1', \ldots, B_s'\}$ B, then $\{B_1', \ldots, B_s'\} \vdash C \rightarrow B$. In particular, if the last line n of Σ has the form

$$n\{H_1, \ldots, H_t\} \quad A \quad \text{with } \{H_1, \ldots, H_t\} \subseteq S, \text{ then}$$

$$(*) \quad \{H_1, \ldots, H_t\} \vdash C \rightarrow A$$

where none of the marked individual variables occur free in $H_1, \ldots, H_t$, A. Now y_r was either marked by an application of ($\forall$I) or by an application of ($\exists$E), and accordingly we have either (i) $C_r = B_r \rightarrow \forall x_r A_r$ or else (ii) $C_r = \exists x_r A_r \rightarrow B_r$. Let $C^{r-1} = C_1 \wedge \ldots \wedge C_{r-1}$ and consider the case (ii) where $C_r = \exists x_r A_r \rightarrow B_r$. On account of (*) we have thus a deduction Δ in the system $\mathcal{S}$ for

$\langle H_1, \ldots, H_t\rangle \vdash C^{r-1} \wedge (\exists x_r A_r \to B_r) \to A$.

We continue this deduction Δ as follows. First of all, applications of (TI) yield:

1. $\langle H_1, \ldots, H_t\rangle \vdash \sim\exists x_r A_r \to (C^{r-1} \to A)$ and
2. $\langle H_1, \ldots, H_t\rangle \vdash B_r \to (C^{r-1} \to A)$

Let z_r be a new individual variable not occurring in Δ. Then there exist unique formulas C', A', H_1', . . . , H_t' such that Subst C^{r-1} x_r/z_r C' and Subst C' z_r/x_r C^{r-1}, Subst A x_r/z_r A' and Subst A' z_r/z_r A, and Subst H_i x_r/z_r H_i' and Subst H_i' z_r/x_r H_i for i = 1, . . . , t.
Hence, applying the rule (FS*) with a substitution of z_r for x_r, we get from step 2:

3. $\langle H_1', \ldots, H_t'\rangle \vdash B_r \to (C' \to A')$

Observe that Subst B_r x_r/z_r B_r, since from Subst A_r x_r/y_r B_r it follows that x_r is not free in B_r (unless $x_r = y_r$, in which case $B_r = A_r$ and steps 3-5 are omitted!). By an application of (FS), substituting x_r for y_r, we obtain from step 3:

4. $\langle H_1', \ldots, H_t'\rangle \vdash A_r \to (C' \to A')$

Note that Subst C' y_r/x_r C'. Indeed, first of all y_r is not free in C^{r-1}, since otherwise there would be y_k with $k<r$ which depends on y_r, contrary to the assumed normed ordering $\langle y_1, \ldots, y_r\rangle$. Since $z_r \neq y_r$ and Subst C^{r-1} x_r/z_r C', y_r is thus not free in C' and, hence, we must have Subst C' y_r/x_r C'. Again, since A occurs in the last line of Σ, y_r is not free in A; from Subst A x_r/z_r A' and $z_r \neq y_r$ it follows that y_r is not free in A' and, hence, Subst A' y_r/x_r A'. Finally, since each H_i, i = 1, . . . , t, is an assumption of the last line of Σ, y_r does not occur free in any H_i, and from Subst H_i x_r/z_r H_i' and $z_r \neq y_r$ it follows that y_r is not free in any H_i' and, thus, Subst H_i' y_r/x_r H_i' for i = 1, . . . , t. Next we apply (AP) to step 4 and obtain:

5. $\langle H_1', \ldots, H_t'\rangle \vdash \exists x_r A_r \to (C' \to A')$

Note that by construction x_r is not free in any of the formulas H_1', . . . , H_t', C', A'. A further application of (FS*), substituting x_r for z_r, yields now from step 5:

6. $\langle H_1, \ldots, H_t\rangle \vdash \exists x_r A_r \to (C^{r-1} \to A)$

From steps 1 and 6 we get by an application of (TI) finally:

7. $\langle H_1, \ldots, H_t\rangle \vdash C^{r-1} \to A$

In an analogous manner we show that in the case (i) where $C_r = B_r \to \forall x_r A_r$, we arrive at a deduction for $\langle H_1, \ldots, H_t\rangle \vdash C^{r-1} \to A$. Repeating this process r times, we can reduce C until we obtain a deduction for $\langle H_1, \ldots, H_t\rangle \vdash$ A and, hence, for S $\vdash$ A.

The two theorems show that the natural deduction system N and the deduction system S generate the same deducibility relations: S $\vdash$ A if and only is S $\rhd$ A, for any set S of formulas and any formula A. Moreover, the proofs of these two theorems indicate how one can construct a deduction in either one of these two systems, given a deduction in the other system.

The deduction system S is sound and strongly complete, as noted in Schneider (1976); a direct, detailed proof for its soundness and strong completeness is given in Schneider (1973). Thus, in view of the equivalence of the two deduction systems N and S, it follows that also the natural deduction system N is sound and strongly complete: A formula A is a consequence of a set S of formulas if and only if A is deducible from S in the natural deduction system N.

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